The Gradient of a Curve

In Rates of Change we estimated gradients of curves by drawing tangents. Differentiation computes them exactly.

Consider the chord joining (x, x^2) and a nearby point $\big(x+h, (x+h)^2\big)$ on $y = x^2$. As h shrinks, the chord becomes the tangent. Find the gradient of the chord, and decide what happens as $h \to 0$.

$$\textit{chord gradient} = \frac{(x+h)^2 - x^2}{h} = \frac{2xh + h^2}{h} = 2x + h \xrightarrow{h \to 0} 2x$$

The gradient of $y = x^2$ at any point is exactly $2x$. The same computation on $y = x^3$ gives $3x^2 + 3xh + h^2 \to 3x^2$.

Definition. The derivative of y with respect to x is the gradient function, written

$$\frac{dy}{dx}, \qquad f'(x), \qquad \text{or} \qquad \frac{d}{dx}\big(\dots\big).$$

Theorem

$$y = kx^n \implies \frac{dy}{dx} = nkx^{n-1} \qquad \text{for any rational } n$$

Constants differentiate to 0, and sums differentiate term by term.

Example

Differentiate:

1. $y = x^7$
2. $y = 5x^4 - 3x^2 + 2x - 11$
3. $y = \frac{1}{2}x^6$

1. $7x^6$ *2.* $20x^3 - 6x + 2$ *3.* $3x^5$

Textbook Exercises: SPS Course 6.1, Gradients Worksheet and Exercise 1

Differentiating Awkward Expressions

The rule only applies to *terms* of the form kx^n. Products, quotients and roots must be rewritten first.

Tip

$$\sqrt{x} = x^{\frac{1}{2}} \qquad \frac{1}{x^3} = x^{-3} \qquad \frac{a + bx^2}{x} = ax^{-1} + bx \qquad (x+2)(x-5) = x^2 - 3x - 10$$

Example

Differentiate:

1. $y = \sqrt{x} + \dfrac{4}{\sqrt{x}}$
2. $y = 3x^2 + 6x^{\frac{1}{3}} + \dfrac{2x^3 - 7}{3\sqrt{x}}$

1. $y = x^{1/2} + 4x^{-1/2} \implies \dfrac{dy}{dx} = \frac{1}{2}x^{-1/2} - 2x^{-3/2}$
2. $\dfrac{2x^3 - 7}{3\sqrt{x}} = \frac{2}{3}x^{5/2} - \frac{7}{3}x^{-1/2}$, so $\dfrac{dy}{dx} = 6x + 2x^{-2/3} + \frac{5}{3}x^{3/2} + \frac{7}{6}x^{-3/2}$

Example (Edexcel C1)

The curve C has equation

$$y = \frac{(x+3)(x-8)}{x}, \quad x > 0.$$

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in its simplest form.

$$y = \frac{x^2 - 5x - 24}{x} = x - 5 - 24x^{-1}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + 24x^{-2} = 1 + \frac{24}{x^2}$$

Example

Find the gradient of $y = 2x - 8\sqrt{x} + 5$ at $x = 4$, and find the point on the curve where the gradient is 0.

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 4x^{-1/2}$. *At* $x = 4$: $2 - 2 = 0$ — *so* $x = 4$ is *the point:* $(4, -3)$.

Textbook Exercises: SPS Course 6.1, Exercise 2A

Tangents and Normals

Fact — At the point $(a, f(a))$ on $y = f(x)$:

- the **tangent** has gradient $f'(a)$;
- the **normal** is perpendicular to the tangent: gradient $-\dfrac{1}{f'(a)}$.

Then use $y - y_1 = m(x - x_1)$.

Example (Edexcel C1)

The curve C has equation $y = x^3 - 2x^2 - x + 3$. The point $P(2, 1)$ lies on C.

1. Show that the equation of the tangent to C at P is $y = 3x - 5$.
2. The tangent to C at a point Q is parallel to the tangent at P. Find the coordinates of Q.

1. $\dfrac{dy}{dx} = 3x^2 - 4x - 1$; *at* $x = 2$: $m = 3$. $y - 1 = 3(x-2) \implies y = 3x - 5$.

2. $3x^2 - 4x - 1 = 3 \implies 3x^2 - 4x - 4 = 0 \implies (3x+2)(x-2) = 0$

$x = -\frac{2}{3}$ *(rejecting* $x = 2$, *which is* P*):* $Q\left(-\frac{2}{3}, \frac{61}{27}\right)$.

Example

Find the equation of the normal to $y = \sqrt{x}$ at the point $(9, 3)$, and the coordinates of the point where this normal meets the curve again.

$\dfrac{dy}{dx} = \frac{1}{2}x^{-1/2}$; at $x = 9$: tangent gradient $\frac{1}{6}$, normal gradient -6.
Normal: $y - 3 = -6(x - 9) \implies y = 57 - 6x$.
Meets curve: $\sqrt{x} = 57 - 6x$. Let $u = \sqrt{x}$: $6u^2 + u - 57 = 0 \implies (u - 3)(6u + 19) = 0$.
$u \geq 0$ gives only $u = 3$, i.e. the point $(9, 3)$ itself: this normal does not meet the curve again.

Textbook Exercises: SPS Course 6.1, Exercise 3

Problems with Parameters

Example (Edexcel C1, adapted)

The curve C has equation $y = 2x^3 + kx^2 + 5x + 6$, where k is a constant.
The point P, where $x = -2$, lies on C. The tangent to C at P is parallel to the line $2y - 18x - 1 = 0$. Find

1. the value of k,
2. the y-coordinate of P,
3. the equation of the tangent at P in the form $ax + by + c = 0$.

Line: $y = 9x + \frac{1}{2}$, *gradient 9.*
1. $\dfrac{dy}{dx} = 6x^2 + 2kx + 5$; *at* $x = -2$: $24 - 4k + 5 = 9 \implies k = 5$.
2. $y = 2(-8) + 5(4) + 5(-2) + 6 = 0$: $P(-2, 0)$.
3. $y = 9(x + 2) \implies 9x - y + 18 = 0$

Example

The curve $y = x^2 + ax + b$ has a tangent at $(2, 7)$ which passes through the origin. Find a and b.

Tangent through $(0,0)$ *and* $(2,7)$: *gradient* $\frac{7}{2}$.
$2x + a = \frac{7}{2}$ *at* $x = 2 \implies a = -\frac{1}{2}$. *On curve:* $4 + 2a + b = 7 \implies b = 4$.

Textbook Exercises: SPS Course 6.1, Exercise 2B

Differentiation and Kinematics

In Rates of Change, velocity and acceleration were gradients read from graphs. Now they are derivatives:

Fact —

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \qquad a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

Example

The displacement, s metres, of a particle from a fixed point O after t seconds is

$$s = 24t^2 - t^3, \qquad 0 \le t \le 20.$$

1. Find expressions for the velocity and the acceleration.
2. Find the times at which the particle is at rest.
3. Find the maximum velocity of the particle.

1. $v = 48t - 3t^2, \quad a = 48 - 6t.$
2. $3t(16 - t) = 0 \implies t = 0$ *or* $t = 16.$
3. *Maximum velocity where* $a = 0$: $t = 8,\ v = 384 - 192 = 192\,\mathrm{m\,s^{-1}}.$

Example

A ball is thrown vertically upwards so that its height above the ground after t seconds is $h = 2 + 15t - 5t^2$ metres.

1. Find the velocity when $t = 1$.
2. Find the greatest height reached.
3. Find the acceleration, and interpret it.

1. $v = 15 - 10t$; at $t = 1$: $5\,\mathrm{m\,s^{-1}}$ (upwards).
2. $v = 0$ at $t = 1.5$: $h = 2 + 22.5 - 11.25 = 13.25\,\mathrm{m}$.
3. $a = -10\,\mathrm{m\,s^{-2}}$: constant downward acceleration (gravity).

Textbook Exercises: SPS Course 6.1, Exercise 2A Q (kinematics parts) and Exercise 6